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Quantum critical behaviour of itinerant ferromagnets

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Abstract. The quantum ferromagnetic transition of itinerant electrons is considered. We give a paedagogical review of recent results which show that zero-temperature soft modes that are commonly neglected invalidate the standard Landau–Ginzburg–Wilson description of this transition. If these modes are taken into account, then the resulting order parameter field theory is non-local in space and time. Nevertheless, both for disordered and for clean systems the critical behaviour has been exactly determined for spatial dimensions $d > 2$ and $d > 1$, respectively. The critical exponents characterizing the paramagnetic-to-ferromagnetic transition are dimensionality-dependent and substantially different both from mean-field critical exponents and from the classical Heisenberg exponents that characterize the transition at finite temperatures. Our results should be easily observable, particularly those for the disordered case, and experiments to check our predictions are proposed.

1. Introduction

Phase transitions that occur in a quantum mechanical system at zero temperature ($T = 0$) as a function of some non-thermal control parameter are called quantum phase transitions. In contrast to their finite-temperature counterparts, which are often referred to as thermal or classical phase transitions, the critical fluctuations with which one has to deal at zero temperature are quantum fluctuations rather than thermal ones and the need for a quantum mechanical treatment of the relevant statistical mechanics makes the theoretical description of quantum phase transitions somewhat different from that of classical ones. However, as Hertz has shown in a seminal paper [1], the basic theoretical concepts that have led to successful descriptions and understanding of thermal transitions work in the quantum case as well.

Experimentally, the zero-temperature behaviour of any material of course cannot be studied directly; furthermore, the most obvious control parameter that drives a system through a quantum transition is often some microscopic coupling strength that is hard to change experimentally. As a result, the dimensionless distance from the critical point, t, which for classical transitions with a transition temperature T_c is given by $t = T/T_c - 1$ and is easy to tune with high accuracy, is much harder to control in the quantum case. However, t is usually dependent on some quantity that can be experimentally controlled, such as the composition of the material. Also, the zero-temperature critical behaviour manifests itself already at low but finite temperatures. Indeed, in a system with a very low thermal transition temperature all but the final asymptotic behaviour in the critical region

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is dominated by quantum effects. The study of quantum phase transitions is therefore far from being of theoretical interest only.

Perhaps the most obvious example of a quantum phase transition is the paramagnet-to-ferromagnet transition of itinerant electrons at $T = 0$ as a function of the exchange interaction between the electronic spins. Early theoretical work [1] on this transition suggested that the critical behaviour in the physical dimensions $d = 2$ and 3 was not dominated by fluctuations and mean-field like, as is the thermal ferromagnetic transition in dimensions $d > 4$. The reason for this is a fundamental feature of quantum statistical mechanics, namely the fact that statics and dynamics are coupled. As a result, a quantum mechanical system in d dimensions is very similar to the corresponding classical system in $d + z$ dimensions, where the so-called dynamical critical exponent z can be thought of as an extra dimensionality that is provided to the system by time or temperature. The $(d + z)$-dimensional space relevant to the statistical mechanics of the quantum system bears some resemblance to $(d + 1)$-dimensional Minkowski space, but z does *not* need to be equal to unity in non-relativistic systems. For clean and disordered itinerant quantum ferromagnets, one finds $z = 3$ and $z = 4$, respectively, in mean-field theory. This appears to reduce the upper critical dimension d_c^+, above which fluctuations are unimportant and simple mean-field theory yields the correct critical behaviour, from $d_c^+ = 4$ in the classical case to $d_c^+ = 1$ and $d_c^+ = 0$, respectively, in the clean and disordered quantum cases. If this were true, then this quantum phase transition would be rather uninteresting from a critical phenomena point of view.

It has been known for some time that, for the case of disordered systems, this conclusion cannot be correct [2]. It is known that, in any system with quenched disorder that undergoes a phase transition, the critical exponent ν that describes the divergence of the correlation length, $\xi \sim t^{-\nu}$ for $t \to 0$, must satisfy the inequality $\nu \geqslant 2/d$ [3]. However, mean-field theory yields $\nu = \frac{1}{2}$, which is incompatible with this inequality for $d < 4$. Technically, this implies that the disorder must be a relevant perturbation with respect to the mean-field fixed point. The mean-field fixed point must therefore be unstable and the phase transition must be governed by some other fixed point that has a correlation length exponent $\nu \geqslant 2/d$.

Such a non-mean-field-like fixed point has recently been discovered and the critical behaviour has been determined exactly for all dimensions $d > 2$ [4]. It was found that both the value $d_c^+ = 0$ for the upper critical dimension and the prediction of mean-field critical behaviour for $d > d_c^+$ were incorrect. Instead, $d_c^+ = 2$ and, although both the quantum fluctuations and the disorder fluctuations are irrelevant with respect to the new fixed point for all $d > d_c^+$, there are two other 'upper critical dimensionalities', $d_c^{++} = 4$ and $d_c^{+++} = 6$. The critical behaviour for $d_c^+ < d < d_c^{+++}$ is governed by a non-standard Gaussian fixed point with non-mean-field-like exponents and only for $d > d_c^{+++}$ does one obtain mean-field exponents. In addition, the clarification of the physics behind this surprising behaviour has led to the conclusion that very similar effects occur in clean systems [5]. In that case, $d_c^+ = 1$ in agreement with the early result, but again the critical behaviour is non-trivial in a range of dimensionalities $d_c^+ < d \leqslant d_c^{++} = 3$, and only for $d > d_c^{++}$ does one obtain mean-field critical behaviour. In addition, we have found that Hertz's $1 - \epsilon$ expansion for the clean case is invalid. This explains an inconsistency between this expansion and an exact exponent relation that was noted earlier by Sachdev [6]. In order to keep our discussion focused, in that which follows we will restrict ourselves to the disordered case, for which the effects are more pronounced, and will only quote results for the clean case when appropriate.

The basic physical reason behind the complicated behaviour above the upper critical dimensionality d_c^+, namely in a regime in parameter space in which the critical behaviour

is not dominated by fluctuations, is simple. According to our general understanding of continuous phase transitions or critical points, in order to understand the critical singularities at any such transition, one must identify all of the slow or soft modes near the critical point and one must make sure that all of these soft modes are properly included in the effective theory for the phase transition. This is obvious, since critical phenomena are effects that occur on very large length and time scales; hence soft modes, whose excitation energies vanish in the limit of long wavelengths and small frequencies, will in general influence the critical behaviour. In previous work on the ferromagnetic transition it was implicitly assumed that the only relevant soft modes are the fluctuations of the order parameter, namely the magnetization. For finite temperatures this is correct. However, at $T = 0$ there are additional soft modes in a disordered electron system, namely diffusive particle–hole excitations that are distinct from the spin density excitations that determine the magnetization. In many-body perturbation theory these modes manifest themselves as products of retarded and advanced Green's functions and in field theory they can be interpreted as the Goldstone modes that result from the spontaneous breaking of the symmetry between retarded and advanced correlation functions, or between positive and negative imaginary frequencies. In a different context, namely the transport theory for disordered electron systems, these diffusive excitations are sometimes referred to as 'diffusons' and 'Cooperons', respectively, and they are responsible for what is known as 'weak localization effects' in disordered electron systems [7]. For our purposes, their most important feature is their spatial long-range nature in the zero-frequency limit. This long-range nature follows immediately from the diffusion equation

$$\left(\partial_t - D\partial_{\boldsymbol{x}}^2\right) f(\boldsymbol{x}, t) = 0 \tag{1.1a}$$

for some diffusive quantity f, with D the diffusion constant. Solving this equation by means of a Fourier–Laplace transform to wavevectors $\boldsymbol{q}$ and complex frequencies z, one obtains in the limit of zero frequency

$$f(\boldsymbol{q}, z = 0) = \frac{1}{D\boldsymbol{q}^2} f(\boldsymbol{q}, t = 0). \tag{1.1b}$$

Long-range static correlations are thus an immediate consequence of the diffusive nature of the density dynamics in disordered systems.

The fact that we are concerned with the zero-frequency or long-time limit is due to the order parameter, namely the magnetization, being a conserved quantity. Since the only way to change the order parameter density locally is to transport this conserved quantity from one region in space to another, in order to develop long-range order over arbitrarily large distances the system needs an infinitely long time. This in turn means that criticality can be attained only if the frequency is taken to zero before the wavenumber. This feature would be lost if there were some type of spin-flip scattering mechanism present and our results hold only in the absence of such processes. For the same reason, they do not apply to quantum antiferromagnets, which show a quite different behaviour [8].

It is important that the long-range static correlations mentioned above are distinct from the order parameter fluctuations. For instance, the latter are soft only at the critical point and in the ordered phase, whereas the former are soft even in the paramagnetic phase and they do not change their nature at the critical point. However, since they couple to the conserved order parameter, they influence the critical behaviour. If one integrates out these diffusive modes in order to obtain an effective theory or Landau–Ginzburg–Wilson (LGW) functional in terms of the order parameter only, then their long-range nature leads to infrared singular integrals, which in turn results in singular vertices in the LGW functional or diverging coupling constants for couplings between the order parameter fluctuations. The

usual LGW philosophy of deriving an effective local field theory entirely in terms of the order parameter field therefore does not lead to a well-behaved field theory in this case. The situation is analogous to a well-known phenomenon in high-energy physics. Suppose that some interaction between, say, fermions, is mediated by the exchange of some other particles, for example gauge bosons of mass M. If the bosons are integrated out, then the resulting theory will be non-renormalizable, that is, it will be ill-behaved on momentum scales larger than the mass M. The non-renormalizable theory corresponds to the order parameter LGW theory, except that in statistical mechanics one runs into infrared problems rather than ultraviolet ones. Nevertheless, it turns out that the critical behaviour in our case can still be determined exactly even after having integrated out the additional soft modes. The point is that the diffusive modes lead to an effective long-range interaction between the order parameter fluctuations that falls off in real space like r^{2-2d}. It is known that in general long-range interactions suppress fluctuation effects [9]. In our case they are strong enough to suppress not only quantum fluctuations but also any remaining disorder fluctuations. The critical behaviour is thus dominated neither by quantum fluctuations (since we work above the upper critical dimension d_c^+) nor by the disorder fluctuations, but rather is given by a simple, albeit non-standard (because of the long-range interactions) Gaussian theory. The resulting Gaussian fixed point allows a correlation length exponent that satisfies $\nu \geqslant 2/d$, as required, and the exponents are dimensionality-dependent for all $d < 6$. In $d = 3$ they are substantially different either from the mean-field exponents or from those for a classical Heisenberg ferromagnet. This has striking observable consequences, as we will discuss.

The outline of this paper is as follows. In section 2 we first discuss some general aspects of itinerant ferromagnets and then we give our results for the critical exponents and for the equation of state near the critical point. Since the purpose of this paper is to give an exposition and discussion of these results that is as non-technical as possible, they will be presented without any derivations. In section 3 we discuss these results as well as several possible experiments that could be performed to test our predicitions. Finally, in section 4 we sketch the derivation of our theoretical results.

2. Results

In order to put the phase transition we are going to consider in perspective, let us first discuss the qualitative phase diagram that one expects for a disordered itinerant electron system in $d = 3$. Let $F_0^a < 0$ be the Fermi-liquid parameter that characterizes the strength of the system's tendency towards ferromagnetism. For $|F_0^a| < 1$ the system is paramagnetic with a spin susceptibility $\chi_s \sim 1/(1 + F_0^a)$, whereas for $|F_0^a| > 1$ the clean Fermi liquid has a ferromagnetic ground state. In figure 1 we show the qualitative phase diagram one expects for a disordered system at $T = 0$ in the F_0^a–λ plane, where λ is some dimensionless measure of the disorder. For $\lambda = 0$, we have the transition from a paramagnetic metal (PM) to a ferromagnetic metal (FM) at $F_0^a = -1$. At small but non-zero λ this transition will occur at somewhat smaller values of $|F_0^a|$, since the disorder effectively increases the spin-triplet electron–electron interaction amplitude and hence $|F_0^a|$. This accounts for the curvature downwards of the PM–FM transition line. At $|F_0^a| = 0$, a metal–insulator transition of Anderson type is known to occur at a critical disorder value λ_c [10]. At non-zero $|F_0^a|$ such a transition from a paramagnetic metal to a paramagnetic insulator (PI) still occurs, albeit it now is what is called an Anderson–Mott transition that occurs at a somewhat larger value of the disorder [11]. The two transition lines will meet at a multicritical point M, and for large values of λ and $|F_0^a|$ one expects a ferromagnetic insulator (FI). The transitions from the FM and PI phases, respectively, to the FI phase have not been studied theoretically,

which is why we denote them by broken lines in figure 1. We will be mostly interested in the phase transition that occurs across the PM–FM transition line at finite disorder, but far away from the metal–insulator transition. However, in section 3 we will return to the remaining regions in this phase diagram.

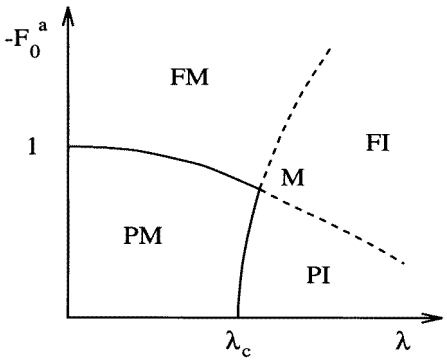

Figure 1. A schematic phase diagram for a $d = 3$ disordered itinerant electron system in the plane spanned by the Landau parameter F_0^a and the disorder λ at $T = 0$. See the text for further explanations.

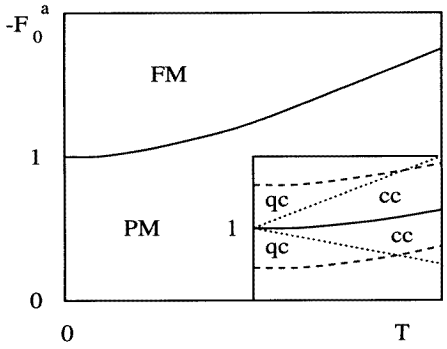

Figure 2. A schematic phase diagram for a disordered itinerant electron system in the plane spanned by the Landau parameter F_0^a and the temperature T. The inset shows the boundary of the critical region (broken line) and the cross over line (dotted line) that separates classical critical behaviour (cc) from quantum critical behaviour (qc).

In figure 2 we show the same phase diagram in the F_0^a–T plane for some value of the disorder $0 < \lambda \ll \lambda_c$. With increasing temperature T, the critical value of $|F_0^a|$ increases, since, in order to achieve long-range order, a larger $|F_0^a|$ is needed to compensate for the disordering effect of the thermal fluctuations. The inset shows schematically the boundary of the critical region (the broken line) and the cross over line between classical and quantum critical behaviour (the dotted line). At any non-zero T, the asymptotic critical behaviour is that of a classical Heisenberg magnet, but at sufficiently low T there is a sizeable region where quantum critical behaviour can be observed.

Our theoretical results for the zero-temperature paramagnet-to-ferromagnet transition can be summarized as follows. Let t be the dimensionless distance from the line separating

the regions PM and FM in figure 1. Then the equation of state, which determines the magnetization m as a function of t and the magnetic field h, can be written

$$tm + m^{d/2} + m^3 = h \tag{2.1}$$

where we have left out all pre-factors of the various terms. Equation (2.1) is valid for all dimensions $d > 2$. Notice the term $m^{d/2}$, which occurs in addition to what otherwise is an ordinary mean-field equation of state. It is a manifestation of the soft particle–hole excitations mentioned in the section 1. For $d < 6$ it dominates the m^3 term, and hence we have for the exponent β, which determines the vanishing of the zero-field magnetization via $m(t, h = 0) \sim t^\beta$,

$$\beta = \begin{cases} 2/(d-2) & \text{for } 2 < d < 6 \\ 1/2 & \text{for } d > 6. \end{cases} \tag{2.2a}$$

Similarly, the exponent δ, defined by $m(t = 0, h) \sim h^{1/\delta}$, is obtained as

$$\delta = \begin{cases} d/2 & \text{for } 2 < d < 6 \\ 3 & \text{for } d > 6. \end{cases} \tag{2.2b}$$

Now let us consider the order parameter field $M(\boldsymbol{x}, t)$ as a function of space and time, namely the field whose average yields the magnetization, $\langle M(\boldsymbol{x}, t)\rangle = m$. Here the angular brackets $\langle \ldots \rangle$ denote a trace with the full statistical operator; that is, they include a quantum mechanical expectation value, a disorder average and at non-zero temperature also a thermal average. We first consider the case of $T = 0$ and Fourier transform to wavevectors $\boldsymbol{q}$ (with modulus $q = |\boldsymbol{q}|$) and frequencies ω. For the order parameter correlation function $G(q, \omega) = \langle M(\boldsymbol{q}, \omega)M(-\boldsymbol{q}, -\omega)\rangle$ we find in the limit of small q and ω

$$G(q, \omega) = \frac{1}{t + q^{d-2} + q^2 - \mathrm{i}\omega/q^2}. \tag{2.3}$$

Here we have again omitted all pre-factors of the terms in the denominator, since they are of no relevance for our discussion. The most interesting feature in equation (2.3) is the term q^{d-2}. It is again an immediate consequence of the additional soft modes discussed in the first section and equation (2.3), like equation (2.1), is valid for $d > 2$. For $q = \omega = 0$, the correlation function G determines the magnetic susceptibility $\chi_m \sim G(q = 0, \omega = 0)$ in zero magnetic field. Hence we have $\chi_m(t) \sim t^{-1} \sim t^{-\gamma}$, where the last relation defines the critical exponent γ. This yields

$$\gamma = 1 \tag{2.4}$$

which is valid for all $d > 2$. γ thus has its usual mean-field value. However, for non-zero q the anomalous q^{d-2} term dominates the usual q^2-dependence for all $d < 4$. The correlation function at zero frequency can then be written

$$G(q, \omega = 0) \sim \frac{1}{1 + (q\xi)^{d-2}} \tag{2.5a}$$

with the correlation length $\xi \sim t^{-1/(d-2)} \sim t^{-\nu}$. For $d > 4$ the q^2 term is dominant and we have for the correlation length exponent ν

$$\nu = \begin{cases} 1/(d-2) & \text{for } 2 < d < 4 \\ 1/2 & \text{for } d > 4. \end{cases} \tag{2.5b}$$

Note that $\nu \geqslant 2/d$, as it must be according to the discussion in section 1. The wavenumber-dependence of G at criticality, namely at $t = 0$, is characterized by the exponent η: $G(q, \omega = 0) \sim q^{-2+\eta}$. From equation (2.3) we obtain

$$\eta = \begin{cases} 4 - d & \text{for } 2 < d < 4 \\ 0 & \text{for } d > 4. \end{cases} \tag{2.6}$$

Finally, consider the correlation function at a wavenumber such that $q\xi = 1$. Then one can write

$$G(q = \xi^{-1}, \omega) \sim \frac{1}{1 - \mathrm{i}\omega\tau} \tag{2.7a}$$

with the relaxation or correlation time $\tau \sim \xi^2/t \sim \xi^{2+1/\nu} \sim \xi^z$, where the last relation defines the dynamical critical exponent z. From equation (2.5*b*) we thus obtain

$$z = \begin{cases} d & \text{for } 2 < d < 4 \\ 4 & \text{for } d > 4. \end{cases} \tag{2.7b}$$

Notice that, with increasing dimensionality d, the exponents ν, η and z 'lock into' their mean-field values at $d = d_c^{++} = 4$, whereas β and δ do so only at $d = d_c^{+++} = 6$. In the special dimensions $d = 4$ and 6 the power-law scaling behaviour quoted above holds only up to additional multiplicative logarithmic dependences on the variables t, h and T. Since these corrections to scaling occur only in unphysical dimensions they are of academic interest only and we refer the interested reader to [4] for details.

The results for the clean case are qualitatively similar, but the anomalous term in the equation of state, equation (2.1), is m^d instead of $m^{d/2}$. This is because the additional soft modes in that case are ballistic instead of diffusive, so their frequency scales with wavenumber like $\omega \sim q$ rather than $\omega \sim q^2$. As a result, the two special dimensions d_c^{++} and d_c^{+++} coincide, and are now $d_c^{++} = 3$, whereas the upper critical dimension proper, above which fluctuations are irrelevant, is $d_c^{+} = 1$. For $1 < d < 3$, the exponent values are $\beta = \nu = 1/(d-2)$, $\delta = z = d$, $\eta = 3 - d$ and $\gamma = 1$. For $d > 3$, all exponents take on their mean-field values as they do in the disordered case for $d > 6$ and in $d = 3$ there are logarithmic corrections to power-law scaling.

We now turn to the behaviour at non-zero temperatures. Then the equation of state acquires temperature corrections, and it is helpful to distinguish between the cases $m \gg T$ and $m \ll T$, with m and T measured in suitable units. Taking into account the leading corrections in either limit, the equation of state reads

$$\begin{aligned} &tm + m^{d/2}\,(1 + T/m) = h \qquad (\text{for } m \gg T) \\ &\left(t + T^{(d-2)/2}\right) m + m^3 = h \qquad (\text{for } T \gg m). \end{aligned} \tag{2.8}$$

Equation (2.8) shows that, for any non-zero temperature, the asymptotic critical behaviour is not given by the quantum critical exponents. Since equation (2.8) takes temperature into account only perturbatively, it correctly describes only the initial deviation from the quantum critical behaviour and approximates the classical critical behaviour by the mean-field result. A full cross over calculation would yield instead the classical Heisenberg critical behaviour in the asymptotic limit. Also, we are considering only the saddle-point contribution to the magnetization. For models with no additional soft modes it has been shown that fluctuations that act as dangerous irrelevant variables introduce another temperature scale that dominates the one obtained from the saddle point [2, 12]. In the present case, however, fluctuations are suppressed by the long-range nature of the effective field theory and the fluctuation

temperature scale is sub-dominant. The behaviour described by equation (2.8) can be summarized by means of the generalized homogeneity law

$$m(t, T, H) = b^{-\beta/\nu} m(tb^{1/\nu}, Tb^{\phi/\nu}, Hb^{\delta\beta/\nu}) \tag{2.9a}$$

where β, ν and δ have the values given above, and b is an arbitrary scale factor. Here

$$\phi = 2\nu \tag{2.9b}$$

is the cross over exponent that describes the deviation from the quantum critical behaviour due to the relevant perturbation provided by the non-zero temperature. The entry $Tb^{\phi/\nu} = Tb^2$ in the scaling function in equation (2.9*a*) reflects the fact that the temperature-dependence of the saddle-point solution is determined by that of the diffusive modes; that is, frequency or temperature scales like $T \sim q^2 \sim b^{-2}$. The critical temperature scale, $T \sim b^{-z}$, would be dominant if it were present, but since the leading behaviour of the magnetization is not determined by critical fluctuations, it is suppressed.

By differentiating equation (2.9*a*) with respect to the magnetic field h, one obtains an analogous homogeneity law for the magnetic susceptibility, χ_m, namely

$$\chi_m(t, T, H) = b^{\gamma/\nu} \chi_m(tb^{1/\nu}, Tb^{\phi/\nu}, Hb^{\delta\beta/\nu}) \tag{2.10a}$$

with

$$\gamma = \beta(\delta - 1) = 1 \tag{2.10b}$$

in agreement with equation (2.4). This result is in agreement with a more direct calculation of χ_m: the same temperature corrections that modify the equation of state, equation (2.8), lead to a replacement of the term q^{d-2} in the denominator of equation (2.3) by $(q^2+T)^{(d-2)/2}$. Since the homogeneous order parameter correlation function determines the spin or order parameter susceptibility, this yields

$$\chi_m(t, T) = \frac{1}{t + T^{1/2\nu}} \tag{2.10c}$$

in agreement with equations (2.10*a*) and (2.10*b*).

Finally, the critical behaviour of the specific heat c_V has been calculated. It is most convenient to discuss the specific heat coefficient, $\gamma_V = \lim_{T\to 0} c_V/T$, which in a Fermi liquid would simply be a constant. Its behaviour at criticality, $t = 0$, is adequately represented by the integral

$$\gamma_V = \int_0^\Lambda \mathrm{d}q \frac{q^{d-1}}{T + q^d + q^4 + h^{1-1/\delta} q^2}. \tag{2.11a}$$

Remarkably, in zero magnetic field, γ_V diverges logarithmically as $T \to 0$ for all dimensions $2 < d < 4$. This can be shown to be a consequence of the dynamical exponent z being exactly equal to the spatial dimensionality d in that range of dimensionalities. If one restores the dependence of γ_V on t, then one obtains a generalized homogeneity law with a logarithmic correction for the leading scaling behaviour of γ_V

$$\gamma_V(t, T, H) = \Theta(4 - d) \ln b + F_\gamma(tb^{1/\nu}, Tb^z, Hb^{\delta\beta/\nu}) \tag{2.11b}$$

Here $\Theta(x)$ denotes the step function and F_γ is an unknown scaling function. Note that γ_V is determined by Gaussian fluctuations and depends on the critical temperature scale; that is, T scales like $t^{\nu z}$ in equation (2.11*b*). This is the leading temperature scale and whenever it is present it dominates the diffusive temperature scale that manifests itself in equations (2.9) and (2.10).

In the clean case, equations (2.9*a*) and (2.10) still hold, if one uses the appropriate exponent values and replaces equation (2.9*b*) by $\phi = \nu$. In equation (2.11*a*), the term q^4

in the denominator of the integrand is replaced by q^3 and consequently the argument of the Θ function in equation (2.11*b*) is $3-d$ rather than $4-d$.

3. Experimental implications and discussion

3.1. Experimental implications

Let us now discuss the experimental implications of the results presented in the preceding section. Obviously, one needs a material that shows a transition from a paramagnetic state to a ferromagnetic one at zero temperature as a function of some experimentally tunable parameter x. Obvious candidates are magnetic alloys of the stoichiometry P_xF_{1-x}, with P a paramagnetic metal and F a ferromagnetic one. Such materials show the desired transition as a function of the composition parameter x; examples include Ni for the ferromagnetic component and Al or Ga for the paramagnetic one [13]. At the critical concentration x_c they also are substantially disordered, but due to the fact that both constituents are metals they are far from any metal–insulator transition. Our theory should therefore be applicable to these systems. The schematic phase diagram at $T=0$ in the T–x plane is shown in figure 3. Notice that this is a realistic phase diagram, as opposed to the 'theoretical' ones in figures 1 and 2. A change in the composition parameter x leads, besides to a change in F_0^a, to many other changes in the microscopic parameters of the system. As x is varied, the system will therefore move on a complicated path in the diagram shown in, say, figure 1. However, since the critical behaviour near the transition is universal, it is independent from the exact path travelled.

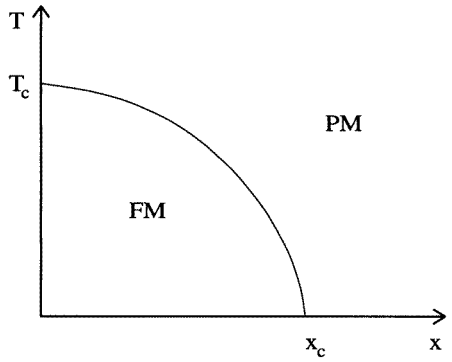

Figure 3. A schematic phase diagram for an alloy of the form P_xF_{1-x}. T_c is the Curie temperature for the pure ferromagnet F and x_c is the critical concentration.

One possible experiment would consist of driving the system at a low, fixed temperature through the transition by changing the composition x. Although this involves the preparation of many samples, this way of probing a quantum phase transition has been used to observe the metal–insulator transition in P-doped Si [14]. It might also be possible to use the stress-tuning technique that has been used for the same purpose [15]. Either way one will cross the transition line along a more or less vertical path in figure 2 and, for a sufficiently low temperature, this path will go through both the classical and the quantum critical region indicated in the inset in figure 2. Due to the large difference between the quantum critical exponents quoted in section 2 and the corresponding exponents for classical Heisenberg magnets, the resulting cross over should be very pronounced and easily observable. For

instance, for $d = 3$ systems our equation (2.2*a*) predicts $\beta = 2$, whereas the value for the thermal transition is $\beta_{class} \approx 0.37$. The resulting cross over in the critical behaviour of the magnetization is schematically shown in figure 4. Alternatively, one could prepare a sample with a value of x that is as close as possible to x_c and measure the magnetic-field-dependence of the magnetization, extrapolated to $T = 0$, to obtain the exponent δ. Again, there is a large difference between our prediction of $\delta = 1.5$ in $d = 3$ and the classical value $\delta_{class} \approx 4.86$.

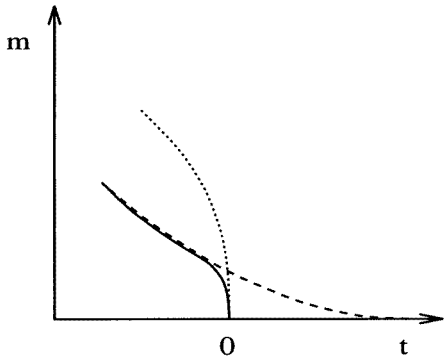

Figure 4. The schematic critical behaviour of the magnetization m at non-zero temperature, showing the cross over from the quantum critical behaviour ($\beta = 2$, broken line) to the classical critical behaviour ($\beta \approx 0.37$, dotted line). Notice that the actual transition is classical in nature.

Yet another possibility is to measure the zero-field magnetic susceptibility as a function both of $t = |x - x_c|$ and of T. Equation (2.10*a*) predicts

$$\chi_m(t, T) = T^{-1/2} f_\chi(T/t^2). \tag{3.1}$$

Here f_χ is a scaling function that has two branches, f_χ^+ for $x > x_c$ and f_χ^- for $x < x_c$. Both branches approach a constant for large values of their argument, $f_\chi^\pm(y \to \infty) = \text{constant}$. For small arguments, we have $f_\chi^+(y \to 0) \sim \sqrt{y}$, whereas f_χ^- diverges at a non-zero value y^* of its argument that signals the classical transition, $f_\chi^-(y \to y^*) \sim (y - y^*)^{-\gamma_{class}}$, with $\gamma_{class} \approx 1.39$ the susceptibility exponent for the classical transition. Our prediction is then that a plot of $\chi_m T^{1/2}$ versus T/t^2 will yield a universal function the shape of which is schematically shown in figure 5. Notice that the exponents are known *exactly*, so the only adjustable parameter for plotting experimental data will be the position of the critical point. This is in sharp contrast to some other quantum phase transitions, especially metal–insulator transitions, for which the exponent values are not even approximately known, which makes scaling plots almost meaningless [16].

Finally, one can consider the low-temperature behaviour of the specific heat. According to equation (2.11*b*), as the temperature is lowered for $x \gtrsim x_c$ the leading temperature-dependence of the specific heat will be

$$c_V(T) \sim T \ln T. \tag{3.2a}$$

At criticality this behaviour will continue to $T = 0$, whereas for $x > x_c$ it will cross over to

$$c_V(T) \sim (\ln t)T. \tag{3.2b}$$

For $x \lesssim x_c$ one will encounter the classical Heisenberg transition at which the specific heat shows a finite cusp (that is the exponent α, defined by $c_V \sim (T - T_c)^{-\alpha}$, is negative).

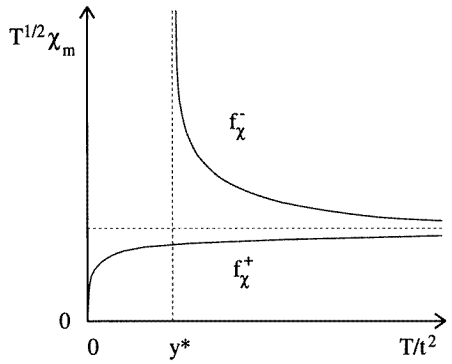

Figure 5. A schematic prediction for a scaling plot of the magnetic suscteptibility.

3.2. Theoretical discussion

There are also various theoretical implications of the results presented in section 2. One aspect is the general message that the usual LGW philosophy must not be applied uncritically to quantum phase transitions, because of the large number of soft modes that exist at zero temperature in a generic system. If any of these couple to the order parameter, then an effective theory entirely in terms of the order parameter will not be well behaved. In the present case we have actually been able to use this to our advantage, since the long-ranged interaction that the additional soft modes induce in the order parameter theory suppresses the disorder fluctuations, which is the reason for the remarkable fact that we are able to determine the critical behaviour of a three-dimensional, disordered system exactly. In general, however, the presence of soft modes in addition to the order parameter fluctuations will call for the derivation of a more complete low-energy effective theory that keeps *all* of the soft modes explicitly.

Another very interesting aspect is a connection between our results on the ferromagnetic transition and a substantial body of literature on a problem that appears in the theory of the metal–insulator transition in interacting disordered electron systems, namely the transition from PM to PI in figure 1. This problem has been known ever since the metal–insulator transition of interacting disordered electrons was first considered and it has led to substantial confusion in that field. Early work on the metal–insulator transition showed that, in two-dimensional systems without impurity spin-flip scattering, the spin-triplet interaction amplitude scaled to large values under renormalization group iterations [17]. This is still true in $d = 2 + \epsilon$ and, since the run-away flow occurs before the metal–insulator transition has been reached, this precluded the theoretical description of the latter in such systems. This problem was interpreted, incorrectly as it turned out later, as a signature of local moment formation in all dimensions [18]. Subsequently, the present authors studied this problem in some detail [19]. We were able to re-sum the perturbation theory explicitly and showed that, at a critical value of the interaction strength, or of the disorder, there is a bulk, thermodynamic phase transition in $d > 2$ that is *not* the metal–insulator transition. Although this ruled out local moments (which would not lead to a phase transition), the physical meaning of this transition was obscure at the time since no order parameter had been identified and its description was entirely in terms of soft diffusion modes. However, the critical exponents obtained are identical to those given in section 2 for the quantum ferromagnetic phase transition and in both cases logarithmic corrections to

scaling are found [20]. Because the exponents in the two cases are identical, we conclude that the transition found earlier by us, whose physical nature was unclear, is actually the ferromagnetic transition. One also concludes that our speculations in [19] about the nature of the ordered phase as an 'incompletely frozen spin phase' with no long-range magnetic order were not correct; this phase is actually the metallic ferromagnetic phase. On the other hand, the techniques used in [19] allowed a determination of the qualitative phase diagram as a function of dimensionality, which our present analysis is not capable of doing. This analysis showed the existence of yet another interesting dimensionality above $d = 2$, which we denote d^*. With the appropriate re-interpretation of the 'incompletely frozen spin phase' as the ferromagnetic phase, the qualitative phase diagram for $2 < d < d^*$ is shown in figure 6. Compared to figure 1, the following happens as d is lowered from $d = 3$. The multicritical point M moves downwards, and at $d = d^*$ it reaches the λ axis. d^* was estimated in [19] to be approximately $d^* = 2.03$. For $d < d^*$, the insulator phase therefore cannot be reached directly from the paramagnetic metal. This explains why in the perturbative renormalization group calculations in $d = 2 + \epsilon$ one necessarily encounters the ferromagnetic transition first and it should finally end the long discussion about the physical meaning of the run-away flow that is encountered in these theories. It also shows that none of these theories is suitable for studying the metal–insulator transition in the absence of spin-flip mechanisms, because they start out in the wrong phase.

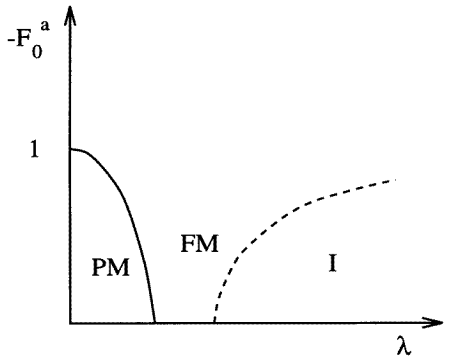

Figure 6. A schematic phase diagram for a disordered itinerant electron system at $T = 0$ close to $d = 2$. The phases shown are the paramagnetic metal (PM), the ferromagnetic metal (FM) and the insulator (I) phase. It is not known whether there is another phase transition within I from a ferromagnetic to a paramagnetic insulator

It should also be pointed out that our earlier theory depended crucially on there being electronic spin conservation. This feature would be lost of there were some type of impurity spin-flip-scattering process. In that case, the soft modes that lead to the long-range order parameter interactions acquire a mass or energy gap and, at sufficiently large scales, the interactions are effectively of short range. The asymptotic critical phenomena in this case are described by a short-range, local order parameter field theory with a random mass, or temperature, term. Such a term is present in the case of a conserved order parameter also, but due to the long-ranged interaction it turns out to be irrelevant with respect to the non-trivial Gaussian fixed point. In the absence of the conservation law, however, the random mass term is relevant with respect to the Gaussian fixed point analogous to the one discussed here. This emphasizes the important role that is played by the order parameter

being conserved in our model. The quantum phase transition in a model in which it is not conserved has been discussed in [8].

We finally discuss why some of our results are in disagreement with Sachdev's general scaling analysis [6] of quantum phase transitions with conserved order parameters. For instance, it follows from our equations (2.10) and (2.11*b*) that the Wilson ratio, defined as $W = (m/H)/(C_V/T)$, diverges at criticality rather than being a universal number as predicted in [6]. Also, for $2 < d < 4$ the function F_γ in equation (2.11*b*), for $t = 0$ and neglecting corrections to scaling, is a function of T/H, in agreement with [6], but for $d > 4$ this is not the case. The reason for this breakdown of general scaling is that we are working above an upper critical dimensionality and hence dangerous irrelevant variables [21] appear that prevent a straightforward application of the results of [6] to the present problem. These dangerous irrelevant variables have to be considered very carefully, on a case by case basis. This caveat is particularly relevant for quantum phase transitions since they tend to have a low upper critical dimension. It is well known that a given irrelevant variable can be dangerous with respect to some observables but not with respect to others. Specifically, in our case there is a dangerous irrelevant variable that affects the leading scaling behaviour of the magnetization, but not that of the specific heat coefficient, which leads to the divergence of the Wilson ratio. This dangerous irrelevant variable is also the reason why the exponents β and δ, which describe the critical behaviour of the magnetization, remain dimensionality-dependent up to $d = 6$, whereas all other exponents 'lock into' their mean-field values already at $d = 4$.

4. A theoretical outline

Here we sketch the derivation of the results that were presented in section 2. We do so for completeness only and will be very brief. A detailed account of the derivation can be found in [4] for the disordered case and in [5] for the clean case.

Hertz [1] has shown how to derive an LGW functional for a quantum ferromagnet. One starts by separating the spin-triplet part of the electron–electron interaction, namely the interaction between spin density fluctuations, from the rest of the action, writing

$$S = S_0 + S_{int}^{(t)} \tag{4.1a}$$

with

$$S_{int}^{(t)} = \frac{\Gamma_t}{2} \int \mathrm{d}x \, \boldsymbol{n}_s(x) \cdot \boldsymbol{n}_s(x). \tag{4.1b}$$

Here $S_{int}^{(t)}$ is the spin-triplet interaction part of the action and S_0 contains all other parts, in particular the electron–electron interaction in all other channels. Γ_t is the spin-triplet interaction amplitude, which is related to the Landau parameter F_0^a used above by $\Gamma_t = -F_0^a/(1 + F_0^a)$, $\boldsymbol{n}_s(x)$ is the electron spin density vector, $x = (\boldsymbol{x}, \tau)$ denotes space and imaginary time, and $\int \mathrm{d}x = \int \mathrm{d}\boldsymbol{x} \int_0^{1/T} \mathrm{d}\tau$. In the critical region near a quantum phase transition, imaginary time scales like a length to the power z and the space–time nature of the integrals in the action accounts for the system's effective dimension $d + z$ that was mentioned in section 1.

Now $S_{int}^{(t)}$ is de-coupled by means of a Hubbard–Stratonovich transformation [1]. The partition function, apart from a non-critical multiplicative constant, can then be written

$$Z = \int D[\boldsymbol{M}] \exp\left(-\Phi[\boldsymbol{M}]\right) \tag{4.2a}$$

with the LGW functional

$$\Phi[\boldsymbol{M}] = \frac{\Gamma_t}{2}\int \mathrm{d}x\, \boldsymbol{M}(x)\cdot\boldsymbol{M}(x) - \ln\left\langle \exp\left[-\Gamma_t \int \mathrm{d}x\, \boldsymbol{M}(x)\cdot\boldsymbol{n}_s(x)\right]\right\rangle_{S_0}. \tag{4.2b}$$

Here $\langle\ldots\rangle_{S_0}$ denotes an average taken with the action S_0. If the LGW functional Φ is formally expanded in powers of $\boldsymbol{M}$, then the term of order $\boldsymbol{M}^n$ obviously has a coefficient that is given by a connected n-point spin-density correlation function of the 'reference system' defined by the action S_0.

At this point we need to remember that our reference system S_0 contains quenched disorder, which has not been averaged over yet. The n-point correlation functions that form the coefficients of the LGW functional therefore still depend explicitly on the particular realization of the randomness in the system. The average over the quenched disorder, which we denote by $\{\ldots\}_{dis}$, requires averaging of the free energy, that is we are interested in $\{\ln Z\}_{dis}$. This is most easily done by means of the replica trick [22]; that is, one writes

$$\{\ln Z\}_{dis} = \lim_{n\to 0}\frac{1}{n}(\{Z^n\}_{dis} - 1) = \lim_{n\to 0}\frac{1}{n}\left[\int\prod_\alpha D[\boldsymbol{M}^\alpha]\left\{\exp\left(-\sum_{\alpha=1}^{n}\Phi^\alpha[\boldsymbol{M}^\alpha]\right)\right\}_{dis} - 1\right] \tag{4.3}$$

where the index α labels n identical replicas of the system. The disorder average is now easily calculated by expanding the exponential in equation (4.3). Upon re-exponentiation, the coefficients in the replicated LGW functional are disorder-averaged correlation functions of the reference system that are cumulants with respect to the disorder average. The Gaussian part of Φ^α is simply

$$\Phi^\alpha_{(2)}[\boldsymbol{M}^\alpha] = \tfrac{1}{2}\int \mathrm{d}x_1\,\mathrm{d}x_2\, \boldsymbol{M}^\alpha(x_1)[\delta(x_1-x_2) - \Gamma_t\chi(x_1-x_2)]\cdot\boldsymbol{M}^\alpha(x_2). \tag{4.4}$$

Here $\chi(x)$ is the disorder averaged spin susceptibility or two-point spin-density correlation function of the reference system. The cubic term, $\Phi^\alpha_{(3)}$, has a coefficient given by the averaged three-point spin-density correlation function. For the quartic term, the cumulant nature of these correlation functions leads to two terms with different replica structures and higher order terms have correspondingly more complicated structures.

The next step is to calculate the spin-density correlation functions for the reference system. It now becomes important that we have kept in our action S_0 the electron–electron interaction in all channels except for the spin-triplet one that has been de-coupled in deriving the LGW functional. At this point our treatment deviates from that of Hertz, who took the reference ensemble to describe non-interacting electrons. This was generally considered an innocent approximation that should not have any qualitative effects. However, this belief was mistaken, since the spin-density correlations of interacting electrons are qualitatively different from those of non-interacting ones. The spin susceptibility can be easily calculated in perturbation theory. The result shows that the static spin susceptibility as a function of the wavenumber q is non-analytical at $q = 0$. For small q it has the form

$$\chi(q) = \text{constant} - q^{d-2} - q^2. \tag{4.5}$$

The non-analyticity is a consequence of the presence of soft particle–hole excitations in the spin-triplet channel and it occurs only in an interacting electron system. That is, the pre-factor of the q^{d-2} term, which we have suppressed in equation (4.5), vanishes for vanishing interaction amplitudes. Renormalization group arguments can then be used to ascertain that this perturbative result indeed represents the exact behaviour of χ in the long-wavelength

limit. If one also considers the frequency-dependence of χ, one obtains the Gaussian part of the LGW functional in the form

$$\Phi^{\alpha}_{(2)}[\boldsymbol{M}] = \tfrac{1}{2}\sum_{\boldsymbol{q}}\sum_{\omega_n} \boldsymbol{M}^{\alpha}(\boldsymbol{q},\omega_n)(t_0 + q^{d-2} + q^2 + |\omega_n|/q^2)\cdot \boldsymbol{M}^{\alpha}(-\boldsymbol{q},-\omega_n) \tag{4.6a}$$

where

$$t_0 = 1 - \Gamma_t \chi_s(\boldsymbol{q}\to 0, \omega_n = 0) \tag{4.6b}$$

is the bare distance from the critical point and the $\omega_n = 2\pi T n$ are bosonic Matsubara frequencies.

The Gaussian theory, equations (4.6), can be analysed using standard renormalization group techniques [23]. Such an analysis reveals the existence of a Gaussian fixed point whose critical properties are the ones given in section 2. The remaining question is whether this fixed point is stable with respect to the higher, non-Gaussian terms in the action. These higher terms also need to be considered in order to obtain the critical behaviour of the magnetization.

A calculation of the higher correlation functions that determine the non-Gaussian vertices of the field theory shows that the non-analyticity that is analogous to the one in the spin susceptibility, equation (4.5), is stronger and results in a divergence of these correlation functions in the zero-frequency, long-wavelength limit. Specifically, the leading behaviour of the n-point spin-density correlation that determines the coefficient of the term of order $\boldsymbol{M}^n$ in the LGW functional, considered at vanishing frequencies as a function of a representative wavenumber q, is

$$\chi^{(n)}(q\to 0) \sim q^{d+2-2n}. \tag{4.7}$$

As a result, the coefficients cannot, as usual, be expanded about zero wavenumber and the theory is non-local. Despite this unpleasant behaviour of the field theory, it is easy to see by power counting that all of these terms except for one are irrelevant with respect to the Gaussian fixed point in all dimensions $d > 2$. The one exception is the quartic cumulant contribution that is the disorder average of the square of the spin susceptibility, which is marginal in $d = 4$, but irrelevant in all other dimensions. This term is physically of interest, since it represents the random mass or random temperature contribution that one would expect in a theory of disordered magnets, that was mentioned in section 3.2 above.

The conclusion from these considerations is that, apart from logarithmic corrections to scaling in certain special dimensions, the Gaussian theory yields the exact critical behaviour and the only remaining question pertains to the form of the equation of state. Since the quartic coefficient $\chi^{(4)}$ is a dangerous irrelevant variable for the magnetization, this requires a scaling interpretation of the infrared divergence of $\chi^{(4)}$. In [4] it has been shown that, for scaling purposes, the wavenumber q in equation (4.7) can be identified with the magnetization $m^{1/2}$. This is physically plausible, since the divergence stems from an integration over soft modes that are rendered massive by an external magnetic field. Since a non-zero magnetization acts physically like a magnetic field, it cuts off the singularity in equation (4.7). With this interpretation of the singular coefficients, the term of order m^n in the saddle-point solution of the LGW theory has the structure $m^{n-1}(m^{1/2})^{d+2-2n} = m^{d/2}$, which leads to the equation of state given in equation (2.1). One might wonder why the magnetic fluctuations in the paramagnetic phase do not also cut off the singularity in equation (4.7) and thus weaken or even destroy the effects discussed above. Although such a cut-off mechanism does exist, it enters the theory only via the fluctuations, which are RG irrelevant with respect to the Gaussian fixed point. It therefore manifests itself only in the corrections to scaling, not in the leading critical behaviour.

Again, all of these arguments can be repeated for the case without disorder. The only changes one encounters pertain to the values of various exponents due to the different character of the soft modes. This leads to the results quoted in section 2.

Acknowledgments

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